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Intermediate Wakimoto modules for affine $\mathfrak{sl}(n+1, \mathbb{C})$

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Abstract

We construct certain boson-type realizations of affine $\mathfrak{sl}(n+1, \mathbb{C})$ that depend on a parameter $0 \leq r \leq n$ such that when $r = 0$ we get a Fock space realization appearing in [6] and when $r = n$ they are the Wakimoto modules described in the work of Feigin and Frenkel [7].

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1. Introduction

Wakimoto modules for affine Lie algebras were introduced by Feigin and Frenkel in [7] by a homological characterization. These modules admit a remarkable boson realization on a Fock space [17] (for $\hat{\mathfrak{sl}}(2)$), [9] (for $\hat{\mathfrak{sl}}(n)$) which plays an important role in the conformal field theory providing a new bosonization rule for the Wess–Zumino–Witten models. Wakimoto modules have a geometric interpretation as certain sheaves on a semi-infinite flag manifold [8]. They belong to the category \mathcal{O} and generically are isomorphic to corresponding Verma modules.

Affine Lie algebras admit Verma-type modules associated with non-standard Borel subalgebras, see [5, 11, 13]. Modules associated with the *natural Borel subalgebra* were first introduced by Jakobsen and Kac in [13]. They were studied in [10] under the name of *imaginary Verma modules*.

A Fock space realization of the imaginary Verma modules for $\hat{\mathfrak{sl}}(2)$ was constructed by Bernard and Felder in [1] and then extended in [6] to the case of $\hat{\mathfrak{sl}}(n)$. These realizations are given generically by certain Wakimoto-type modules.

The main motivation for our work was a problem of finding suitable boson-type realizations for all Verma-type modules over $\widehat{\mathfrak{sl}}(n+1)$. In theorem 3.1 we construct such realizations, *intermediate Wakimoto modules*, for a series of generic Verma-type modules depending on the parameter $0 \leq r \leq n$. If $r = n$ this construction coincides with the boson realization of Wakimoto modules in [7]. On the other hand, when $r = 0$ the obtained representation gives a Fock space realization described in [6]. One difficulty that arises in the study of Verma-type modules that are not induced from a standard Borel subalgebra is that certain of their weight spaces are infinite dimensional. On the other hand, the structure of representations that have infinite-dimensional weight spaces is an important problem that appears naturally in other contexts. Besides appearing in the representation theory of infinite-dimensional Heisenberg Lie algebras such representations also arise in the work of [3, 4]. Intermediate Wakimoto modules are another family of representations with certain weight spaces being infinite dimensional. We plan to discuss their detailed structure in a subsequent paper using the construction given in this paper. See the concluding remarks for the potential usefulness of our result.

2. Preliminaries

Fix a positive integer n , $0 \leq r \leq n$, $\gamma \in \mathbb{C}^*$. Set $k = \gamma^2 - (r+1)$. Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ and let $E^{ij}, i, j = 1, \dots, n+1$, be the standard basis for $\mathfrak{gl}(n+1, \mathbb{C})$. Set $H_i := E^{ii} - E^{i+1,i+1}, E_i := E^{i,i+1}, F_i := E^{i+1,i}$ which is a basis for $\mathfrak{sl}(n+1, \mathbb{C})$. Furthermore we denote the Killing form by $(X|Y) = \text{tr}(XY)$ and $X_m = t^m \otimes X$ for $X, Y \in \mathfrak{g}$ and $m \in \mathbb{Z}$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a base for Δ^+ , the positive set of roots for \mathfrak{g} , such that $H_i = \check{\alpha}_i$ and let Δ_r be the root system with base $\{\alpha_1, \dots, \alpha_r\}$ ($\Delta_r = \emptyset$, if $r = 0$) of the Lie subalgebra $\mathfrak{g}_r = \mathfrak{sl}(r+1, \mathbb{C})$. A Cartan subalgebra \mathfrak{h} (respectively \mathfrak{h}_r) of \mathfrak{g} (respectively \mathfrak{g}_r) is spanned by $H_i, i = 1, \dots, n$ (respectively $i = 1, \dots, r$), and set $\mathfrak{h}_0 = 0$.

For any Lie algebra \mathfrak{a} , let $L(\mathfrak{a}) = \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra of \mathfrak{a} . Then $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(n+1, \mathbb{C}) = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{g}}_r = L(\mathfrak{g}_r) \oplus \mathbb{C}c \oplus \mathbb{C}d$ are the associated affine Kac–Moody algebras with $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{h}}_r = \mathfrak{h}_r \oplus \mathbb{C}c \oplus \mathbb{C}d$ respectively.

The algebra $\hat{\mathfrak{g}}$ has generators $E_{im}, F_{im}, H_{im}, i = 1, \dots, n, m \in \mathbb{Z}$, and central element c with the product

$$[X_m, Y_n] = t^{m+n}[X, Y] + \delta_{m+n,0}m(X|Y)c.$$

2.1. Oscillator algebras

Let $\hat{\mathfrak{a}}$ be the infinite-dimensional Heisenberg algebra with generators $a_{ij,m}, a_{ij,m}^*$ and $\mathbf{1}, 1 \leq i \leq j \leq n$ and $m \in \mathbb{Z}$, subject to the relations

$$\begin{aligned} [a_{ij,m}, a_{kl,n}] &= [a_{ij,m}^*, a_{kl,n}^*] = 0 & [a_{ij,m}, a_{kl,n}^*] &= \delta_{ik}\delta_{jl}\delta_{m+n,0}\mathbf{1} \\ [a_{ij,m}, \mathbf{1}] &= [a_{ij,m}^*, \mathbf{1}] = 0. \end{aligned}$$

Such an algebra has a representation $\tilde{\rho} : \hat{\mathfrak{a}} \rightarrow \mathfrak{gl}(\mathbb{C}[\mathbf{x}])$ where

$$\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_{ij,m} | i, j, m \in \mathbb{Z}, 1 \leq i \leq j \leq n]$$

denotes the algebra over \mathbb{C} generated by the indeterminates $x_{ij,m}$ and $\tilde{\rho}$ is defined by

$$\begin{aligned} \tilde{\rho}(a_{ij,m}) &:= \begin{cases} \partial/\partial x_{ij,m} & \text{if } m \geq 0 \text{ and } j \leq r \\ x_{ij,m} & \text{otherwise} \end{cases} \\ \tilde{\rho}(a_{ij,m}^*) &:= \begin{cases} x_{ij,-m} & \text{if } m \leq 0 \text{ and } j \leq r \\ -\partial/\partial x_{ij,-m} & \text{otherwise} \end{cases} \end{aligned}$$

and $\tilde{\rho}(\mathbf{1}) = 1$. In this case $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$ -module generated by $1 =: |0\rangle$, where

$$a_{ij,m}|0\rangle = 0 \quad m \geq 0 \text{ and } j \leq r \quad a_{ij,m}^*|0\rangle = 0 \quad m > 0 \text{ or } j > r.$$

Let $\hat{\mathfrak{a}}_r$ denote the subalgebra generated by $a_{ij,m}$ and $a_{ij,m}^*$ and $\mathbf{1}$, where $1 \leq i \leq j \leq r$ and $m \in \mathbb{Z}$. If $r = 0$, we set $\hat{\mathfrak{a}}_r = 0$.

Let $A_n = ((\alpha_i|\alpha_j))$ be the Cartan matrix for $\mathfrak{sl}(n+1, \mathbb{C})$ and let \mathfrak{B} be the matrix whose entries are

$$\mathfrak{B}_{ij} := (\alpha_i|\alpha_j) \left(\gamma^2 - \delta_{i>r} \delta_{j>r} (r+1) + \frac{r}{2} \delta_{i,r+1} \delta_{j,r+1} \right)$$

where

$$\delta_{i>r} = \begin{cases} 1 & \text{if } i > r \\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$\mathfrak{B} := \gamma^2 A_n - (r+1) \begin{pmatrix} 0 & 0 \\ 0 & A_{n-r} \end{pmatrix} + r E_{r+1,r+1}.$$

We also have the Heisenberg Lie algebra $\hat{\mathfrak{b}}$ with generators $b_{im}, 1 \leq i \leq n, m \in \mathbb{Z}, \mathbf{1}$, and relations $[b_{im}, b_{jp}] = m \mathfrak{B}_{ij} \delta_{m+p,0} \mathbf{1}$ and $[b_{im}, \mathbf{1}] = 0$.

For each $1 \leq i \leq n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ has a representation $\rho_\lambda : \hat{\mathfrak{b}} \rightarrow \mathfrak{gl}(\mathbb{C}[\mathbf{y}])$ where

$$\mathbb{C}[\mathbf{y}] := \mathbb{C}[y_{i,m} | i, m \in \mathbb{N}^*, 1 \leq i \leq n]$$

and ρ_λ is defined on $\mathbb{C}[\mathbf{y}]$ by

$$\rho_\lambda(b_{i0}) = \lambda_i \quad \rho_\lambda(b_{i,-m}) = \mathbf{e}_i \cdot \mathbf{y}_m \quad \rho_\lambda(b_{im}) = m \mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m} \quad \text{for } m > 0$$

and $\rho_\lambda(\mathbf{1}) = 1$. Here

$$\mathbf{y}_m = (y_{1m}, \dots, y_{nm}) \quad \frac{\partial}{\partial \mathbf{y}_m} = \left(\frac{\partial}{\partial y_{1m}}, \dots, \frac{\partial}{\partial y_{nm}} \right)$$

and \mathbf{e}_i are vectors in \mathbb{C}^n such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$, where \cdot means the usual dot product.

Note that since \mathfrak{B}_{ij} is symmetric, it is orthogonally diagonalizable (i.e. there exists an orthogonal matrix P such that $P^t \mathfrak{B} P$ is a diagonal matrix), and hence we can find vectors \mathbf{e}_i in \mathbb{C}^n such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$. In fact for $m > 0$ and $n < 0$ we get

$$\begin{aligned} [b_{im}, b_{jn}] &= \left[m \mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m}, \mathbf{e}_j \cdot \mathbf{y}_{-n} \right] \\ &= m \sum_{k,l} \left[e_{ik} \frac{\partial}{\partial y_{km}}, e_{jl} y_{l,-n} \right] \\ &= m \delta_{m+n,0} \sum_k e_{ik} e_{jk} = m \delta_{m+n,0} \mathfrak{B}_{ij}. \end{aligned}$$

(See also [9].)

2.2. Formal distributions

We need some more notation that will simplify some of the arguments later. This notation follows roughly [14] and [15]: a *formal distribution* is an expression of the form

$$a(z, w, \dots) = \sum_{m,n,\dots \in \mathbb{Z}} a_{m,n,\dots} z^m w^n$$

where the $a_{m,n,\dots}$ lie in some fixed vector space V and z, w, \dots are formal variables. We define $\partial a(z) = \partial_z a(z) = \sum_n n a_n z^{n-1}$. We also have expansion about zero: there are two canonical embeddings of vector spaces $\iota_{z,w} : \mathbb{C}(z-w) \rightarrow \mathbb{C}[[z,w]]$ and $\iota_{w,z} : \mathbb{C}(z-w) \rightarrow \mathbb{C}[[z,w]]$ where $\iota_{z,w}(a(z,w))$ is a formal Laurent series expansion in z^{-1} and $-\iota_{w,z}(a(z,w))$ is a formal Laurent series expansion in z . The *formal delta function* $\delta(z-w)$ is the formal distribution

$$\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{w}\right)^n = \iota_{z,w} \left(\frac{1}{z-w}\right) - \iota_{w,z} \left(\frac{1}{z-w}\right).$$

For any sequence of elements $\{a_m\}_{m \in \mathbb{Z}}$ in the ring $\text{End}(V)$, V being a vector space, the formal distribution

$$a(z) := \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$$

is called a *field*, if for any $v \in V$, $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then we set

$$a(z)_- := \sum_{m \geq 0} a_m z^{-m-1} \quad \text{and} \quad a(z)_+ := \sum_{m < 0} a_m z^{-m-1}.$$

In particular

$$\delta(z-w)_- = \iota_{z,w} \left(\frac{1}{z-w}\right) \quad \delta(z-w)_+ = -\iota_{w,z} \left(\frac{1}{z-w}\right).$$

Note that

$$-\partial_z \delta(z-w) = \partial_w \delta(z-w) = \iota_{z,w} \left(\frac{1}{(z-w)^2}\right) - \iota_{w,z} \left(\frac{1}{(z-w)^2}\right).$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} : a_m b_n : z^{-m-1} w^{-n-1} =: a(z)b(w) := a(z)_+ b(w) + b(w) a(z)_-. \tag{2.1}$$

For any $1 \leq i \leq j \leq n$, we define

$$a_{ij}^*(z) = \sum_{n \in \mathbb{Z}} a_{ij,n}^* z^{-n} \quad a_{ij}(z) = \sum_{n \in \mathbb{Z}} a_{ij,n} z^{-n-1}$$

and

$$b_i(z) = \sum_{n \in \mathbb{Z}} b_{in} z^{-n-1}.$$

In this case

$$[b_i(z), b_j(w)] = \mathfrak{B}_{ij} \partial_w \delta(z-w) \quad [a_{ij}(z), a_{kl}^*(w)] = \delta_{ik} \delta_{jl} \mathbf{1} \delta(z-w).$$

Observe that $a_{ij}(z)$ for $j > r$ is not a field whereas $a_{ij}^*(z)$ is always a field. We will call $a_{ij}(z)$ (resp. $a_{ij}^*(z)$) a *pure creation* (resp. *annihilation*) operator if $j > r$. Set

$$a_{ij}(z)_+ = a_{ij}(z) \quad a_{ij}(z)_- = 0 \quad a_{ij}^*(z)_+ = 0 \quad a_{ij}^*(z)_- = a_{ij}^*(z)$$

if $j > r$.

Now we should point out that while $a^1(z_1) \cdots a^m(z_m)$ is always defined as a formal series, we will only define $a(z)b(z) ::= \lim_{w \rightarrow z} a(z)b(w)$ for certain pairs $(a(z), b(w))$. For example

$$: a_{ij}(z)a_{kl}^*(z) := \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} : a_{ij,n} a_{kl,m-n}^* : \right) z^{-m-1}$$

is well defined as an element in $\text{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}] [[z, z^{-1}]])$ for all $l > r$ (as $\tilde{\rho}(a_{kl,m}^*) := -\partial/\partial x_{kl,-m}$ for $l > r$) or if both $l \leq r$ and $j \leq r$ (see also the remarks after theorem 3.1).

Then one defines recursively

$$: a^1(z_1) \cdots a^k(z_k) := : a^1(z_1) (: a^2(z_2) (: \cdots : a^{k-1}(z_{k-1}) a^k(z_k) :) \cdots :) :$$

while the normal ordered product

$$: a^1(z) \cdots a^k(z) := \lim_{z_1, z_2, \dots, z_k \rightarrow z} : a^1(z_1) (: a^2(z_2) (: \cdots : a^{k-1}(z_{k-1}) a^k(z_k) :) \cdots :) :$$

will only be defined for certain k -tuples (a^1, \dots, a^k) .

Let

$$[ab] = a(z)b(w) - : a(z)b(w) := [a(z)_-, b(w)] \tag{2.2}$$

(half of $[a(z), b(w)]$) denote the contraction of any two formal distributions $a(z)$ and $b(w)$ where $a(z), b(z)$ are free fields or pure creation or annihilation operators. For example if $j, l \leq r$, then

$$[a_{ij} a_{kl}^*] = \sum_{m \geq 0} \delta_{ik} \delta_{jl} z^{-m-1} w^m = \delta_{i,k} \delta_{j,l} \delta_-(z-w) = \delta_{ik} \delta_{jl} t_{z,w} \left(\frac{1}{z-w} \right) \tag{2.3}$$

$$[a_{kl}^* a_{ij}] = - \sum_{n < 0} \delta_{ik} \delta_{jl} z^n w^{-n-1} = -\delta_{i,k} \delta_{j,l} \delta_+(w-z) = \delta_{ik} \delta_{jl} t_{z,w} \left(\frac{1}{w-z} \right). \tag{2.4}$$

If $l > r$, then

$$[a_{ij} a_{kl}^*] = [a_{ij}(z)_-, a_{kl}^*(w)] = 0 \tag{2.5}$$

$$[a_{kl}^* a_{ij}] = [a_{kl}^*(z)_-, a_{ij}(w)] = -\delta_{i,k} \delta_{j,l} \delta(w-z). \tag{2.6}$$

Theorem 2.1 (Wick's theorem, [2], [12] or [14]). *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra $\text{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$, satisfying*

- (1) $[[a^i(z)b^j(w)], c^k(x)_\pm] = [[a^i b^j], c^k(x)_\pm] = 0$, for all i, j, k and $c^k(x) = a^k(z)$ or $c^k(x) = b^k(w)$.
- (2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ for all i and j .
- (3) The products

$$[a^{i_1} b^{j_1}] \cdots [a^{i_s} b^{j_s}] : a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w) :_{(i_1, \dots, i_s; j_1, \dots, j_s)}$$

have coefficients in $\text{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$ for all subsets $\{i_1, \dots, i_s\} \subset \{1, \dots, M\}, \{j_1, \dots, j_s\} \subset \{1, \dots, N\}$. Here the subscript $(i_1, \dots, i_s; j_1, \dots, j_s)$ means that those factors $a^i(z), b^j(w)$ with indices $i \in \{i_1, \dots, i_s\}, j \in \{j_1, \dots, j_s\}$ are to be omitted from the product $a^1 \cdots a^M b^1 \cdots b^N$ and when $s = 0$ we do not omit any factors.

Then

$$: a^1(z) \cdots a^M(z) :: b^1(w) \cdots b^N(w) := \sum_{s=0}^{\min(M,N)} \sum_{i_1 < \cdots < i_s, j_1 \neq \cdots \neq j_s} [a^{i_1} b^{j_1}] \cdots [a^{i_s} b^{j_s}] : a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w) :_{(i_1, \dots, i_s; j_1, \dots, j_s)}.$$

Proof. Although it is essentially the same proof as in [12] or [14], we will repeat the argument here for the convenience of the reader.

To simplify notation we will write $a^i = a^i(z_i)$ and $b^j = b^j(w_j)$ below, hoping that it will not cause confusion. Moreover we define $[a^k b^k] = 0$ when $k > \min\{M, N\}$.

The conclusion is true for $M = N = 1$ as follows from the definition of the contraction, (2.2). Suppose now $N > 1$ and $M = 1$. Then by hypotheses 1 and 2, and induction

$$\begin{aligned} a : b^1 \cdots b^N &:= ab^1_+ : b^2 \cdots b^N : + a : b^2 \cdots b^N : b^1_- \\ &= +b^1_+ : b^2 \cdots b^N : + [ab^1] : b^2 \cdots b^N : + b^1_+ a_- : b^2 \cdots b^N : + a : b^2 \cdots b^N : b^1_- \\ &= a_+ b^1_+ : b^2 \cdots b^N : + [ab^1] : b^2 \cdots b^N : + \sum_{j=2}^N [ab^j] b^1_+ : b^2 \cdots \widehat{b^j} \cdots b^N : \\ &\quad + b^1_+ : b^2 \cdots b^N : a_- + a_+ : b^2 \cdots b^N : b^1_- \\ &\quad + \sum_{j=2}^N [ab^j] : b^2 \cdots \widehat{b^j} \cdots b^N : b^1_+ + : b^2 \cdots b^N : b^1_- a_- \\ &= a_+ : b^1 \cdots b^N : + [ab^1] : b^2 \cdots b^N : \\ &\quad + \sum_{j=2}^N [ab^j] : b^1 \cdots \widehat{b^j} \cdots b^N : + : b^1 \cdots b^N : a_- \\ &= : ab^1 \cdots b^N : + \sum_{j=1}^N [ab^j] : b^1 \cdots \widehat{b^j} \cdots b^N : . \end{aligned}$$

The hat above a factor means omit that factor. Note that at each step we are combining summations where all of the coefficients in each summand are in $\text{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$ when z_1, \dots, z_M get replaced by z .

A similar argument proves the result for $M > 1$ and $N = 1$.

Now let us assume that M and N are greater than 1. Then using hypotheses 1, 2 and induction we get

$$\begin{aligned} \sum_{s \geq 0} \sum_{\substack{1 \leq i_1 < \cdots < i_s, \\ j_1 \neq \cdots \neq j_s}} [a^{i_1} b^{j_1}] \cdots [a^{i_s} b^{j_s}] : a^1 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \\ &= a^1_+ \sum_{s \geq 0} \sum_{\substack{2 \leq i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} [a^{i_1} b^{j_1}] \cdots [a^{i_s} b^{j_s}] : a^2 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \\ &\quad + \sum_{s \geq 0} \sum_{\substack{2 \leq i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} [a^{i_1} b^{j_1}] \cdots [a^{i_s} b^{j_s}] : a^2 \cdots a^M b^1 \cdots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} a^1_- \\ &\quad + \sum_{j=1}^N [a^1 b^j] : a^2 \cdots a^M : : b^1 \cdots \widehat{b^j} \cdots b^N : \end{aligned}$$

$$\begin{aligned}
 &= a_+^1 \sum_{s \geq 0} \sum_{\substack{2 \leq i_1 < \dots < i_s \\ j_1 \neq \dots \neq j_s}} [a^{i_1} b^{j_1}] \dots [a^{i_s} b^{j_s}] : a^2 \dots a^M b^1 \dots b^N :_{(i_1, \dots, i_s; j_1, \dots, j_s)} \\
 &\quad + : a^2 \dots a^M :: a_-^1 b^1 \dots b^N : + \sum_{j=1}^N [a^1 b^j] : a^2 \dots a^M :: b^1 \dots \widehat{b^j} \dots b^N : \\
 &= a_+^1 : a^2 \dots a^M :: b^1 \dots b^N : + : a^2 \dots a^M : a_-^1 : b^1 \dots b^N : \\
 &= : a^1 \dots a^M :: b^1 \dots b^N : .
 \end{aligned}$$

Since every step involves combining sums that are (by induction assumed to be) well defined when z_1, \dots, z_M and w_1, \dots, w_N are replaced by z and w respectively, so is the resulting product. This completes the proof of the theorem. \square

We will also need the following two results.

Theorem 2.2 (Taylor’s theorem, [14], 2.4.3). *Let $a(z)$ be a formal distribution. Then in the region $|z - w| < |w|$*

$$a(z) = \sum_{j=0}^{\infty} \partial_w^{(j)} a(w) (z - w)^j. \tag{2.7}$$

Theorem 2.3 ([14], theorem 2.3.2). *Let $a(z)$ and $b(z)$ be formal distributions with coefficients in the associative algebra $\text{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$. The following are equivalent:*

- (i) $[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z - w) c^j(w)$, where $c^j(w) \in \text{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}] [[w, w^{-1}]])$.
- (ii) $[ab] = \sum_{j=0}^{N-1} \iota_{z,w} \left(\frac{1}{(z - w)^{j+1}} \right) c^j(w)$.

In other words the singular part of the operator product expansion

$$[ab] = \sum_{j=0}^{N-1} \iota_{z,w} \left(\frac{1}{(z - w)^{j+1}} \right) c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. One writes

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z - w)^{j+1}}.$$

For example

$$b_i(z)b_j(w) \sim \frac{\delta_{ij}}{(z - w)^2}.$$

2.3. Verma type modules

For a Lie algebra \mathfrak{a} we denote by $U(\mathfrak{a})$ the universal enveloping algebra of \mathfrak{a} .

Let \mathfrak{g}_α be a root subspace of \mathfrak{g} corresponding to a root α , $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ a Cartan decomposition of \mathfrak{g} . Denote also $\mathfrak{n}_r^\pm = \mathfrak{n}^\pm \cap \mathfrak{g}_r$, $\mathfrak{n}^+(r) = \mathfrak{n}^+ \setminus \mathfrak{n}_r^+$,

$$\bar{B}_r = L(\mathfrak{n}^+(r)) \oplus (\mathfrak{n}_r^+ \otimes \mathbb{C}[t]) \oplus ((\mathfrak{n}_r^- \oplus \mathfrak{h}) \otimes \mathbb{C}[t]).$$

Then $B_r = \bar{B}_r \oplus \hat{\mathfrak{h}}$ is a Borel subalgebra of $\hat{\mathfrak{g}}$ for any $0 \leq r \leq n$.

Fix $\tilde{\lambda} \in \hat{\mathfrak{h}}^*$ and consider a $\hat{\mathfrak{g}}$ -module

$$M_r(\tilde{\lambda}) = U(\hat{\mathfrak{g}}) \otimes_{U(B_r)} \mathbb{C}v_{\tilde{\lambda}}$$

where $\bar{B}_r v_{\tilde{\lambda}} = 0$ and $h v_{\tilde{\lambda}} = \tilde{\lambda}(h)v_{\tilde{\lambda}}$ for all $h \in \hat{\mathfrak{h}}$.

Module $M_r(\tilde{\lambda})$ is a particular case of Verma-type modules studied in [5, 11]. When $r = n$ it gives a usual Verma module construction. If $r = 0$ we get an imaginary Verma module.

Let $\tilde{\lambda}_r = \tilde{\lambda}|_{\hat{\mathfrak{h}}_r}$. Verma-type module $M_r(\tilde{\lambda})$ contains a $\hat{\mathfrak{g}}_r$ -submodule $M(\tilde{\lambda}_r) = U(\hat{\mathfrak{g}}_r)(1 \otimes v_{\tilde{\lambda}})$ which is isomorphic to a usual Verma module for $\hat{\mathfrak{g}}_r$.

Theorem 2.4 ([5, 11]). *Let $\tilde{\lambda}(c) \neq 0$. Then the submodule structure of $M_r(\tilde{\lambda})$ is completely determined by the submodule structure of $M(\tilde{\lambda}_r)$. In particular, $M_r(\tilde{\lambda})$ is irreducible if $M(\tilde{\lambda}_r)$ is irreducible.*

3. Intermediate Wakimoto modules

Define

$$\begin{aligned} E_i(z) &= \sum_{n \in \mathbb{Z}} E_{in} z^{-n-1} & F_i(z) &= \sum_{n \in \mathbb{Z}} F_{in} z^{-n-1} \\ H_i(z) &= \sum_{n \in \mathbb{Z}} H_{in} z^{-n-1} & 1 \leq i \leq n. \end{aligned}$$

The defining relations between the generators of $\hat{\mathfrak{g}}$ can be written as follows,

- (R1) $[H_i(z), H_j(w)] = (\alpha_i | \alpha_j) c \partial_w \delta(w - z)$
- (R2) $[H_i(z), E_j(w)] = (\alpha_i | \alpha_j) E_j(z) \delta(w - z)$
- (R3) $[H_i(z), F_j(w)] = -(\alpha_i | \alpha_j) F_j(z) \delta(w - z)$
- (R4) $[E_i(z), F_j(w)] = \delta_{i,j} (H_i(z) \delta(w - z) + c \partial_w \delta(w - z))$
- (R5) $[F_i(z), F_j(w)] = [E_i(z), E_j(w)] = 0$ if $(\alpha_i | \alpha_j) \neq -1$
- (R6) $[F_i(z_1), F_i(z_2), F_j(w)] = [E_i(z_1), E_i(z_2), E_j(w)] = 0$ if $(\alpha_i | \alpha_j) = -1$

where $[X, Y, Z] := [X, [Y, Z]]$ is the Engel bracket for any three operators X, Y, Z .

Recall that $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$ -module with respect to the representation $\tilde{\rho}$ and $\mathbb{C}[\mathbf{y}]$ is a $\hat{\mathfrak{b}}$ -module with respect to ρ_λ . The main result of the paper is the following theorem where we define a representation

$$\rho : \hat{\mathfrak{g}} \rightarrow \mathfrak{gl}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]).$$

We use the notation $\rho(X_m) := \rho(X)_m$, for $X \in \mathfrak{g}$.

Theorem 3.1. Let $\lambda \in \mathfrak{H}^*$ and set $\lambda_i = \lambda(H_i)$. The generating functions

$$\begin{aligned} \rho(F_i)(z) &= a_{ii} + \sum_{j=i+1}^n a_{ij}a_{i+1,j}^* \\ \rho(H_i)(z) &= 2 : a_{ii}a_{ii}^* : + \sum_{j=1}^{i-1} (: a_{ji}a_{ji}^* : - : a_{j,i-1}a_{j,i-1}^* :) \\ &\quad + \sum_{j=i+1}^n (: a_{ij}a_{ij}^* : - : a_{i+1,j}a_{i+1,j}^* :) + b_i \\ \rho(E_i)(z) &=: a_{ii}^* \left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^* - \sum_{k=1}^i a_{ki}a_{ki}^* \right) : + \sum_{k=i+1}^n a_{i+1,k}a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^* \\ &\quad - a_{ii}^*b_i - (\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2)\partial a_{ii}^* \\ \rho(c) &= \gamma^2 - (r+1) \end{aligned}$$

define an action of the generators $E_{im}, F_{im}, H_{im}, i = 1, \dots, n, m \in \mathbb{Z}$ and c , on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$. In the above a_{ij}, a_{ij}^* and b_i denote $a_{ij}(z), a_{ij}^*(z)$ and $b_i(z)$ respectively.

Remark 3.2. One can see that the normal ordered products $: a_{ij}(z)a_{ij}^*(z) :$ are all well defined and thus so is $\rho(H_i)(z)$. Moreover a careful analysis of the other formal distributions shows that they too have coefficients in $\text{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$. For example

$$\sum_{k=i+1}^n a_{i+1,k}a_{ik}^*$$

has summand $a_{i+1,k}a_{ik}^*$ that is certainly well defined if $k \leq r$ and for $k > r$ we have $\tilde{\rho}(a_{ik,m}^*) := -\partial/\partial x_{ik,-m}$ for all $m \in \mathbb{Z}$. Moreover the summation

$$\sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^*$$

is well defined also for $i \leq r$ and for $i > r$ we have $\tilde{\rho}(a_{ki,m}^*) := -\partial/\partial x_{ki,-m}$.

Proof. The rather tedious proof is obtained by a routine application of Wick’s theorem, Taylor’s theorem and theorem 2.3 above. We have left the details in the appendix. \square

Theorem 3.1 defines a boson-type realization of $\hat{\mathfrak{sl}}(n+1, \mathbb{C})$ and a module structure on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ that depends on the parameter $0 \leq r \leq n$. We will call such a module an *intermediate Wakimoto module* and denote it by $W_{n,r}(\lambda, \gamma)$. The intermediate Wakimoto modules $W_{n,r}(\lambda, \gamma)$ have the property that the subalgebra \tilde{B}_r annihilates the vector $1 \otimes 1 \in \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}], h(1 \otimes 1) = \lambda(h)(1 \otimes 1)$ for all $h \in \mathfrak{H}$ and $c(1 \otimes 1) = (\gamma^2 - (r+1))(1 \otimes 1)$. Consider the $\hat{\mathfrak{g}}_r$ -submodule $W = U(\hat{\mathfrak{g}}_r)(1 \otimes 1) \simeq W_{r,r}(\lambda, \gamma)$ of $W_{n,r}(\lambda, \gamma)$. Then W is isomorphic to the Wakimoto module $W_{\lambda(r), \tilde{\gamma}}$ ([9]) where $\lambda(r) = \lambda|_{\mathfrak{H}_r}, \tilde{\gamma} = \gamma^2 - (r+1)$.

Consider $\tilde{\lambda} \in \hat{\mathfrak{H}}^*$ such that $\tilde{\lambda}|_{\mathfrak{H}_r} = \lambda, \tilde{\lambda}(c) = \gamma^2 - (r+1)$, a Verma-type module $M_r(\tilde{\lambda})$ and its $\hat{\mathfrak{g}}_r$ -submodule $M(\tilde{\lambda}_r)$. Suppose that $M(\tilde{\lambda}_r)$ is irreducible. In this case the Wakimoto module $W_{\lambda(r), \tilde{\gamma}}$ is isomorphic to $M(\tilde{\lambda}_r)$. Let $\tilde{W} = U(\hat{\mathfrak{g}})W_{\lambda(r), \tilde{\gamma}}$ and assume that $\lambda(c) \neq 0$. Then by theorem 2.4 the module $M_r(\tilde{\lambda})$ is irreducible and therefore isomorphic to \tilde{W} . Hence theorem 3.1 also provides a boson-type realization for generic Verma-type modules.

4. Concluding remarks

We gave a realization of the intermediate Wakimoto module for $\hat{\mathfrak{sl}}(n + 1, \mathbb{C})$. We list below related problems and planned future work:

- (1) We believe that generically Verma-type modules and intermediate Wakimoto modules are isomorphic (the same way as generic Verma modules are isomorphic to classical Wakimoto modules). Preliminary calculations using (A.3) below give a proof of this for $\hat{\mathfrak{sl}}(n + 1, \mathbb{C})$ when $r = n - 1$. This should be explored further.
- (2) Verma-type modules have a complicated structure when the centre c acts as zero (see for example [10]). The realization above given in theorem 3.1 yields information about the structure of these modules at least in the case of $\hat{\mathfrak{sl}}(2, \mathbb{C})$. These calculations are inspired by work done in [9] when $c = -\check{h}$ is at the singular hyperplane.
- (3) A similar realization must exist for all Verma-type modules over $\hat{\mathfrak{sl}}(n + 1, \mathbb{C})$ and other affine Lie algebras. It would be of interest to us if one could give a characterization and proof of the existence of intermediate Wakimoto modules using semi-infinite flag manifolds and their cohomology (see [8, 16]).

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Appendix: Proof of theorem 3.1

Set

$$\mathcal{H}_i(z) := 2 : a_{ii}a_{ii}^* : + \sum_{j=1}^{i-1} (: a_{ji}a_{ji}^* : - : a_{j,i-1}a_{j,i-1}^* :) + \sum_{j=i+1}^n (: a_{ij}a_{ij}^* : - : a_{i+1,j}a_{i+1,j}^* :).$$

In the above a_{ij} and a_{ij}^* denote $a_{ij}(z)$, $a_{ij}^*(z)$ respectively.

For any $\alpha \in \Delta^+$ we can find unique $1 \leq k \leq l \leq n$ such that

$$\alpha = \alpha_k + \dots + \alpha_l.$$

Set $a_\alpha := a_{kl}$ and $a_\alpha^* := a_{kl}^*$. Observe that

$$(\alpha_i | \alpha) = \sum_{j=k}^l (\alpha_i | \alpha_j) = (2\delta_{ik}\delta_{il} + \delta_{k<i}(\delta_{il} - \delta_{i-1,l}) + \delta_{l>i}(\delta_{ik} - \delta_{i+1,k})).$$

Moreover we have

$$[a_\alpha a_\beta^*] = \begin{cases} \delta_{\alpha,\beta} t_{z,w} \left(\frac{1}{z-w} \right) & \text{if } \alpha, \beta \in \Delta_r^+ \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

$$[a_\alpha^* a_\beta] = \begin{cases} -\delta_{\alpha,\beta} t_{z,w} \left(\frac{1}{z-w} \right) & \text{if } \alpha, \beta \in \Delta_r^+ \\ -\delta_{\alpha,\beta} \delta(w-z) & \text{otherwise.} \end{cases} \tag{A.2}$$

Since this is the case we can rewrite

$$\mathcal{H}_i(z) := \sum_{\alpha \in \Delta^+} (\alpha_i | \alpha) : a_\alpha a_\alpha^* : .$$

Lemma 5.1. For $1 \leq i \leq j \leq n$, $\alpha, \beta \in \Delta^+$,

$$\begin{aligned}
 [\mathcal{H}_i(z), a_\alpha(w)] &= -\delta_{1 \leq i \leq r} \delta_{\alpha \in \Delta_r^+} (\alpha_i | \alpha) a_\alpha(z) \delta(z - w) \\
 [\mathcal{H}_i(z), a_\alpha^*(w)] &= \delta_{1 \leq i \leq r} \delta_{\alpha \in \Delta_r^+} (\alpha_i | \alpha) a_\alpha^*(z) \delta(z - w) \\
 [\mathcal{H}_i(z), \partial_w a_\alpha^*(w)] &= \delta_{1 \leq i \leq r} \delta_{\alpha \in \Delta_r^+} (\alpha_i | \alpha) a_\alpha^*(z) \partial_w \delta(z - w) \\
 [\mathcal{H}_i(z), \mathcal{H}_j(w)] &= -(\alpha_i | \alpha_j) \left((1 - \delta_{i>r} \delta_{j>r})(r + 1) + \frac{r}{2} \delta_{i,r+1} \delta_{j,r+1} \right) \partial_w \delta(z - w) \\
 [\mathcal{H}_i(z), : a_\alpha(w) a_\beta^*(w) a_\gamma^*(w) :] &= (\alpha_i | \beta + \gamma - \alpha) : a_\alpha(w) a_\beta^*(w) a_\gamma^*(w) : \delta(z - w) \\
 &\quad - \delta_{\alpha \in \Delta_r^+} (\alpha_i | \alpha) (\delta_{\alpha, \beta} a_\gamma^*(w) + \delta_{\alpha, \gamma} a_\beta^*(w)) \partial_w \delta(z - w) \\
 [\mathcal{H}_{\alpha_i}(z), : a_\alpha(w) a_\beta(w) a_\gamma^*(w) :] &= (\alpha_i | \gamma - \alpha - \beta) : a_\alpha(w) a_\beta(w) a_\gamma^*(w) : \delta(z - w) \\
 &\quad - \delta_{\gamma \in \Delta_r^+} (\alpha_i | \gamma) (\delta_{\gamma, \beta} a_\alpha(w) + \delta_{\alpha, \gamma} a_\beta(w)) \partial_w \delta(z - w).
 \end{aligned}$$

Lemma 5.2 (R1).

$$[\rho(H_i)(z), \rho(H_j)(w)] = (\alpha_i | \alpha_j) \rho(c) \partial_w \delta(z - w).$$

Proof. We use lemma 5.1 in the following calculations:

$$\begin{aligned}
 [\rho(H_i)(z), \rho(H_j)(w)] &= [\mathcal{H}_i(z) + b_i(z), \mathcal{H}_j(z) + b_j(z)] \\
 &= \left(-(\alpha_i | \alpha_j) \left((1 - \delta_{i>r} \delta_{j>r})(r + 1) + \frac{r}{2} \delta_{i,r+1} \delta_{j,r+1} \right) \right. \\
 &\quad \left. + (\alpha_i | \alpha_j) \left(\gamma^2 - \delta_{i>r} \delta_{j>r}(r + 1) + \frac{r}{2} \delta_{i,r+1} \delta_{j,r+1} \right) \right) \partial_w \delta(z - w) \\
 &= (\alpha_i | \alpha_j) \rho(c) \partial_w \delta(z - w). \quad \square
 \end{aligned}$$

Lemma 5.3 (R2).

$$[\rho(H_i)(z), \rho(E_j)(w)] = (\alpha_i | \alpha_j) \rho(E_j)(z) \delta(z/w).$$

Proof. We will use lemma 5.1 repeatedly:

$$\begin{aligned}
 &[\mathcal{H}_i(z), \rho(E_j)(w)] \\
 &= [\mathcal{H}_i(z), : a_{jj}^* \left(\sum_{k=1}^{j-1} a_{k,j-1} a_{k,j-1}^* - \sum_{k=1}^j a_{kj} a_{kj}^* \right) : + \sum_{k=j+1}^n a_{j+1,k} a_{jk}^* \\
 &\quad - \sum_{k=1}^{j-1} a_{k,j-1} a_{kj}^* - a_{jj}^* b_j - (\delta_{j>r}(r + 1) + \delta_{j \leq r}(j + 1) - \gamma^2) \partial a_{jj}^*] \\
 &= \sum_{k=1}^{j-1} ((\alpha_i | \alpha_j) : a_{jj}^* a_{k,j-1} a_{k,j-1}^* : \delta(z - w) - \delta_{j-1 \leq r} (\alpha_i | \alpha_k \\
 &\quad + \dots + \alpha_{j-1}) a_{jj}^*(w) \partial_w \delta(z - w)) - \sum_{k=1}^j ((\alpha_i | \alpha_j) : a_{jj}^* a_{kj} a_{kj}^* : \delta(z - w) \\
 &\quad - \delta_{j \leq r} (\alpha_i | \alpha_k + \dots + \alpha_j) (\delta_{jk} a_{kk}^*(w) + a_{jj}^*(w)) \partial_w \delta(z - w)) \\
 &\quad + (\alpha_i | \alpha_j) \sum_{k=j+1}^n a_{j+1,k}(z) a_{jk}^*(w) \delta(z - w) - (\alpha_i | \alpha_j)
 \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{k=1}^{j-1} a_{k,j-1}(z)a_{kj}^*(w)\delta(z-w) - (\alpha_i|\alpha_j)a_{jj}^*(z)b_j(w)\delta(z-w) \\
 & - (\alpha_i|\alpha_j)(\delta_{j>r}(r+1) + \delta_{j\leq r}(j+1) - \gamma^2)a_{jj}^*(z)\partial_w\delta(z-w) \\
 = & (\alpha_i|\alpha_j)\left(: a_{jj}^* \left(\sum_{k=1}^{j-1} a_{k,j-1}a_{k,j-1}^* - \sum_{k=1}^j a_{kj}a_{kj}^* \right) : \right. \\
 & \left. + \sum_{k=j+1}^n a_{j+1,k}a_{jk}^* - \sum_{k=1}^{j-1} a_{k,j-1}a_{kj}^* - a_{jj}^*b_j \right)\delta(z-w) \\
 & + \sum_{k=1}^j \delta_{1\leq j\leq r}(\alpha_i|\alpha_k + \dots + \alpha_j)(\delta_{jk}a_{kk}^*(w) + a_{jj}^*(w))\partial_w\delta(z-w) \\
 & - \sum_{k=1}^{j-1} (\delta_{1\leq j-1\leq r}(\alpha_i|\alpha_k + \dots + \alpha_{j-1})a_{jj}^*(w)\partial_w\delta(z-w)) \\
 & - (\alpha_i|\alpha_j)(\delta_{j>r}(r+1) + \delta_{j\leq r}(j+1) - \gamma^2)a_{jj}^*(z)\partial_w\delta(z-w).
 \end{aligned}$$

The last term of $\rho(H_i)(z)$ gives us

$$[b_i(z), \rho(E_j)(w)] = -\mathfrak{B}_{ij}a_{jj}(w)\partial_w\delta(z-w).$$

Putting these computations together we get

$$[\rho(H_i)(z), \rho(E_j)(w)] = (\alpha_i|\alpha_j)\rho(E_j)(w)\delta(z-w).$$

□

Lemma 5.4 (R3).

$$[\rho(H_i)(z), \rho(F_j)(w)] = -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w).$$

Proof. The proof follows from lemma 5.1:

$$\begin{aligned}
 & [\rho(H_i)(z), \rho(F_j)(w)] \\
 = & [\mathcal{H}_i(z), \rho(F_j)(w)] = [\mathcal{H}_i(z), a_{jj}(w) + \sum_{k=j+1}^n a_{jk}(w)a_{j+1,k}^*(w)] \\
 = & -(\alpha_i|\alpha_j)a_{jj}(w)\delta(z-w) + \sum_{k=j+1}^n [\mathcal{H}_i(z), a_{jk}(w)]a_{j+1,k}^*(w) \\
 & + \sum_{k=j+1}^n a_{jk}(w)[\mathcal{H}_i(z), a_{j+1,k}^*(w)] \\
 = & \left(-(\alpha_i|\alpha_j)a_{jj}(w) - \sum_{k=j+1}^n (\alpha_i|\alpha_j + \dots + \alpha_k)a_{jk}(z)a_{j+1,k}^*(w) \right. \\
 & \left. + \sum_{k=j+1}^n (\alpha_i|\alpha_{j+1} + \dots + \alpha_l)a_{jk}(w)a_{j+1,k}^*(z) \right)\delta(z-w) \\
 = & -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w).
 \end{aligned}$$

□

Lemma 5.5 (R4).

$$[\rho(E_i)(z), \rho(F_j)(w)] = \delta_{i,j}(\rho(H_i)(z))\delta(z-w) + \rho(c)\partial_w\delta(z-w)$$

Proof. First we take $i = j$. Now for the convenience of the reader we recall that $\rho(E_i)(z)$ is equal to

$$\begin{aligned} &: a_{ii}^* \left(\sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - \sum_{k=1}^i a_{ki} a_{ki}^* \right) : + \sum_{k=i+1}^n a_{i+1,k} a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1} a_{ki}^* \\ &\quad - a_{ii}^* b_i - (\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2) \partial a_{ii}^* \end{aligned}$$

and thus the first summand of $\rho(F_i)(w) = a_{ii} + \sum_{l=i+1}^n a_{il} a_{i+1,l}^*$ brackets with $\rho(E_i)(z)$ to give us

$$\begin{aligned} &\left(2 : a_{ii}(z) a_{ii}^*(z) : - : \sum_{k=1}^{i-1} (a_{k,i-1} a_{k,i-1}^* - a_{ki} a_{ki}^*) : + b_i(z) \right) \delta(z-w) \\ &\quad + (\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2) \partial_z \delta(z-w). \end{aligned}$$

The second summation in $\rho(F_i)(w)$ contributes

$$\begin{aligned} &\sum_{l=i+1}^n [\rho(E_i)(z), a_{il}(w) a_{i+1,l}^*(w)] \\ &= \sum_{l=i+1}^n \left[\sum_{k=i+1}^n a_{i+1,k}(z) a_{ik}^*(z), a_{il}(w) a_{i+1,l}^*(w) \right] \\ &= \sum_{l=i+1}^n (a_{il}(z) a_{il}^*(z) - a_{i+1,l}(z) a_{i+1,l}^*(z)) \delta(z-w) - \delta_{i+1\leq r}(r-i) \partial_w \delta(z-w). \end{aligned}$$

Adding these two summations, we arrive at the desired result.

Now consider the case $|i - j| \geq 1$. Then $\rho(F_j)(w)$ acts as $a_{jj} + \sum_{l=j+1}^n a_{jl} a_{j+1,l}^*$. First we have

$$\begin{aligned} [E_i(z), a_{jj}(w)] &= \left[: a_{ii}^* \left(\sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - \sum_{k=1}^i a_{ki} a_{ki}^* \right) : + \sum_{k=i+1}^n a_{i+1,k} a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1} a_{ki}^* \right. \\ &= \left[: a_{ii}^* \left(\sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - \sum_{k=1}^i a_{ki} a_{ki}^* \right) : , a_{jj}(w) \right] \\ &= -\delta_{j,i-1} a_{i-1,i-1}(z) a_{i,i}^*(z) \delta(z-w) \end{aligned}$$

by lemma 5.1(d). Next we have

$$\begin{aligned} &\left[E_i(z), \sum_{l=j+1}^n a_{jl}(w) a_{j+1,l}^*(w) \right] \\ &= \left[: a_{ii}^* \left(\sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - \sum_{k=1}^i a_{ki} a_{ki}^* \right) : + \sum_{k=i+1}^n a_{i+1,k} a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1} a_{ki}^* \right. \\ &\quad \left. - a_{ii}^* b_i + (\gamma^2 - \delta_{i+1\leq r}(i+1)) \partial a_{ii}^*, \sum_{l=j+1}^n a_{jl}(w) a_{j+1,l}^*(w) \right] \end{aligned}$$

$$\begin{aligned}
&= \left[: a_{ii}^* \left(\sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - \sum_{k=1}^i a_{ki} a_{ki}^* \right) : - \sum_{k=1}^{i-1} a_{k,i-1} a_{ki}^*, \sum_{l=j+1}^n a_{jl}(w) a_{j+1,l}^*(w) \right] \\
&\quad \text{by lemma 5.1(a)} \\
&= \delta_{j,i-1} a_{i-1,i-1}(z) a_{ii}^*(z) \delta(z-w) \quad \text{by lemma 5.1(b) and (c)}.
\end{aligned}$$

Adding the last two calculations finishes the proof of this lemma. \square

We are now left with the Serre relations:

Lemma 5.6 (R5/R6).

$$\begin{aligned}
[\rho(F_i)(z), \rho(F_j)(w)] &= [\rho(E_i)(z), \rho(E_j)(w)] = 0 \quad \text{if } (\alpha_i | \alpha_j) \neq -1 \\
[\rho(F_i)(z_1), \rho(F_i)(z_2), \rho(F_j)(w)] &= [\rho(E_i)(z_1), \rho(E_i)(z_2), \rho(E_j)(w)] = 0 \\
&\quad \text{if } (\alpha_i | \alpha_j) = -1.
\end{aligned}$$

Proof. Let us check the relations for $\rho(F_i)$. (The Serre relations were already known to hold true for the F_i , see [9], but we provide a proof for the convenience of the reader.) A straightforward calculation shows

$$\left[a_{ii}(z), a_{jj} + \sum_{l=j+1}^n a_{jl} a_{j+1,l}^* \right] = \delta_{i,j+1} a_{j,j+1}(w) \delta(z-w).$$

Moreover

$$\begin{aligned}
&\left[\sum_{k=i+1}^n a_{ik}(z) a_{i+1,k}^*(z), a_{jj}(w) + \sum_{l=j+1}^n a_{jl}(w) a_{j+1,l}^*(w) \right] \\
&= \left(\delta_{i,j+1} \sum_{k=j+2}^n a_{jk}(z) a_{j+2,k}^*(z) - \delta_{j,i+1} \sum_{k=i+2}^n a_{ik}(z) a_{i+2,k}^*(z) - \delta_{j,i+1} a_{i,i+1}(z) \right) \\
&\quad \times \delta(z-w)
\end{aligned}$$

by lemma 5.1(a). Thus

$$\begin{aligned}
[\rho(F_i)(z), \rho(F_j)(w)] &= (\delta_{i,j+1} a_{j,j+1}(w) - \delta_{j,i+1} a_{i,i+1}(z)) \delta(z-w) \\
&\quad + \left(\delta_{i,j+1} \sum_{k=j+2}^n a_{jk}(z) a_{j+2,k}^*(z) - \delta_{j,i+1} \sum_{k=i+2}^n a_{ik}(z) a_{i+2,k}^*(z) \right) \delta(z-w). \quad (\text{A.3})
\end{aligned}$$

Note the above is zero if $|i-j| \neq 1$ which is precisely when $(\alpha_i | \alpha_j) \neq -1$. As the first index in a_{kl} (resp. a_{kl}^*) in $\rho(F_i)(z)$ is i (resp. $i+1$) we also get

$$[\rho(F_i)(z_1), \rho(F_i)(z_1), \rho(F_j)(w)] = 0.$$

This completes the proof of the relations R5 and R6 for $\rho(F_i)(z)$. The Serre relations, R5 and R6, for $\rho(E_i)(z)$ are quite tedious and can be found at the first author's Web page³. \square

³ <http://math.cofc.edu/faculty/cox/papers/>

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